On the Finite–Temperature Generalization of the C-theorem and the Interplay between Classical and Quantum Fluctuations

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Abstract. The behavior of the finite-temperature C-function, defined by Neto and Fradkin [Nucl. Phys. B **400**, 525 (1993)], is analyzed within a d-dimensional exactly solvable lattice model, recently proposed by Vojta [Phys. Rev. B **53**, 710 (1996)], which is of the same universality class as the quantum nonlinear O(n) sigma model in the limit $n \to \infty$. The scaling functions of C for the cases $d = 1$ (absence of long-range order), $d = 2$ (existence of a quantum critical point), $d = 4$ (existence of a line of finite temperature critical points that ends up with a quantum critical point) are derived and analyzed. The locations of regions where C is monotonically increasing (which depend significantly on d) are exactly determined. The results are interpreted within the finite-size scaling theory that has to be modified for $d = 4$.

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1. INTRODUCTION

The original Zamolodchikov's C-theorem is related to zero-temperature systems. It establishes the existence of a dimensionless function C of the coupling constants with monotonic properties along the renormalization group trajectories [1]. The assumptions presented in the proof are related with the energy-momentum conservation, the rotational and translational symmetries, and positivity in a two dimensional quantum field theory. The behavior of the C-function reflects the role the quantum fluctuations and it is useful in determining the qualitative futures of the theory away from the criticality. At the fixed points it takes the value of the central charge of the corresponding conformal field theory. Since the basic assumptions underlying the C-theorem are not specific to two dimensions only, a considerable interest exists in generalization of the Zamolodchikov's result for dimensionalities different from two as well as for nonzero temperatures [2] - [6]. Earlier efforts (see [2] and references therein) have been devoted to finding a version of the C-theorem valid in four dimensions. There the approach was based on a careful investigation of the form of the trace of the energy-momentum tensor, written in terms of finite local composite operators. Despite of being able to write expressions for the Zamolodchikov's equations for the C-function, similar to the case of two dimensions, it turns out that it is not possible to demonstrate the monotonicity property. Let us note also the fact that the 3-d analog of central charge [7] is not equivalent to the universal number characterizing the size dependence of the free energy at the critical point [4] (which is always the case in 2-d conformal field theory). This fact indicates that a straightforward generalization of the Zamolodchikov's C-theorem is not to be expected for a general d. See also Ref. [5], where different approaches to the problem have been offered and where it was shown that no direct relations exist between the "finite temperature C-theorem" and the Zamolodchikov's C-theorem at zero temperature. In the present study an approach, proposed by Netto and Fradkin [3] (in some sense a thermodynamic one), for finding a candidate for C-function will be considered. In [3] the following dimensionless function is defined

$$C\left(\beta, g, a|d\right) = -\beta^{d+1} \frac{v^d}{n(d)} \left[f\left(\beta, g, a|d\right) - f\left(\infty, g, a|d\right)\right], \qquad (1.1)$$

where β is the inverse temperature ($\beta = 1/T$) with the Boltzman constant $k_B = 1$, g is a set of dimensionful coupling constants, $f\left(\infty, g, a|d\right) \equiv E_0\left(g, a|d\right)$ is the zero-temperature energy density, i.e. the energy of the "infinite" in the inverse temperature system, $f\left(\beta, g, a|d\right)$ is the full free energy density (per unit volume) of the system, and a is the characteristic length scale of the lattice. Here $n(d)$ is a positive real number (which depends only on the dimensionality d of the system) and v is the characteristic velocity (e.g. the velocity of quasiparticles) in the system. The function $C\left(\beta, g, a|d\right)$ is considered to be the d-dimensional nonzero temperature extension of the Zamolodchikov's C-

function. It is supposed to be *positive*, and, in the regions where the quantum fluctuations dominate, a *monotonically increasing function of the temperature*. In [3] the numbers $n(d) = \Gamma((d+1)/2)\zeta(d+1)/\pi^{(d+1)/2}$ for bosons ($\zeta(x)$ is the Riemann dzeta function, $\Gamma(x)$ is the gamma function) and $n(d) = \Gamma((d+1)/2)\zeta(d+1)(2-2^{1-d})/\pi^{(d+1)/2}$ for fermions have been suggested. Obviously, the exact choice of $n(d)$ does not effect the monotonicity properties of the C-function. Functions analogous to the one defined in (1.1) have been discussed also in Refs. [4] - [6].

For a general d the existence of phase transitions in the system, as well as the interplay between the quantum and classical fluctuations, makes the analysis of the behavior of the C-function difficult from a general point of view. That is why, for any $d \neq 1$ the properties of C have been considered on the examples of concrete models: the free massive filed theories (for any d) [3], the Ising model in a transverse field (for $d = 1$) [3] and the quantum nonlinear sigma model (QNLσM) in the limit $N \to \infty$ and $d = 2$ [3], [4]. (Recently, the value of the C-function at the critical point as a function of d , $1 < d < 3$, has been calculated in [6] for that model).

In the present article we will consider the d-dependence of the monotonicity property of the C-function within the framework of an exactly solvable lattice model. We will explicitly demonstrate the crucial role that the existence of a quantum ($T = 0$) and/or classical ($T \neq 0$) critical points plays for the behavior of C in different regions of the phase diagram in the plane temperature - parameter controlling the quantum fluctuations.

The article is organized as follows. In Section II we briefly describe the model and in Section III present the basic exact analytical expressions for the free energy of the bulk system at nonzero temperature. Section IV contains the analysis of the behavior of the C-function in dimensions $1, 2$ and 4 in different regimes of the parametric space of the model. Section V presents a finite-size scaling interpretation of the results. The article closes with concluding remarks given in Section VI.

2.   The Model

The model we consider describes a magnetic ordering due to the interaction of quantum spins. The Hamiltonian has the following form [8]

$$\mathcal{H} = \frac{1}{2}g\sum_{\ell}\mathcal{P}_\ell^2 - \frac{1}{2}\sum_{\ell\ell'}J_{\ell\ell'}\mathcal{S}_\ell\mathcal{S}_{\ell'} + \frac{\mu}{2}\sum_{\ell}\mathcal{S}_\ell^2, \tag{2.1}$$

where $\mathcal{S}_\ell$ are spin operators at site ℓ, the operators $\mathcal{P}_\ell$ are "conjugated" momenta (i.e. $[\mathcal{S}_\ell, \mathcal{S}_{\ell'}] = 0$, $[\mathcal{P}_\ell, \mathcal{P}_{\ell'}] = 0$, and $[\mathcal{P}_\ell, \mathcal{S}_{\ell'}] = i\delta_{\ell\ell'}$, with $\hbar = 1$), the coupling constants $J_{\ell,\ell'} = J$ are between nearest neighbors only, the coupling constant g is introduced so as to measure the strength of the quantum fluctuations (below it will be called quantum parameter), and, finally, the spherical field μ is introduced to insure the fulfillment of

the constraint $\langle \mathcal{S}_\ell^2 \rangle = 1$. Here $\langle \cdots \rangle$ denotes the standard thermodynamic average taken with $\mathcal{H}$.

In the thermodynamic limit the reduced free energy $\tilde{f}_\infty (\beta, g|d) = f_\infty (\beta, g|d) / \sqrt{gJ}$ takes the form [9]

$$\lambda \tilde{f}_\infty (t, \lambda|d) = \sup_\phi \left\{ \frac{t}{(2\pi)^d} \int_{-\pi}^{\pi} dq_1 \ldots \int_{-\pi}^{\pi} dq_d \right. \tag{2.2}$$

$$\left. \times \ln \left[2\sinh \left(\frac{\lambda}{2t} \sqrt{\phi + 2 \sum_{i=1}^{d} (1 - \cos q_i)} \right) \right] - \frac{1}{2}\phi \right\} - d,$$

where we have introduced the notations: $\lambda = \sqrt{g/J}$ is the normalized quantum parameter, $t = \frac{T}{J}$ is the normalized temperature and $\phi = \frac{\mu}{J} - 2d$ is the shifted spherical field. The supremum is attained at a solution of the mean–spherical constraint that reads

$$1 = \frac{t}{(2\pi)^d} \sum_{m=-\infty}^{\infty} \int_{-\pi}^{\pi} dq_1 \ldots \int_{-\pi}^{\pi} dq_d \frac{1}{\phi + 2 \sum_{i=1}^{d} (1 - \cos q_i) + b^2 m^2}, \tag{2.3}$$

where $b = 2\pi t/\lambda$.

Eqs. (2.3) and (2.3) provide the basis for studying the critical behavior of the model under consideration.

The critical behavior and some finite size properties of this model have been considered in [10], [11] for $1 < d < 3$. Below we present a brief sketch of the derivation of the bulk free energy for $d = 1, 2, 4$ at low temperatures.

3.   The free energy at low temperatures

By using the identities:

$$\ln \frac{\sinh b}{\sinh a} = \frac{1}{2} \sum_{m=-\infty}^{\infty} \ln \frac{b^2 + \pi^2 m^2}{a^2 + \pi^2 m^2}, \tag{3.1}$$

where $ab > 0$, a, b are arbitrary real numbers,

$$\ln (a + b) = \ln a + \int_0^\infty \exp (-ax) (1 - \exp (-bx)) \frac{dx}{x}, \tag{3.2}$$

where $a > 0$, $a + b > 0$, and the Jacobi identity after some algebra at low temperatures ($\frac{\lambda}{t} \gg 1$) the expression for the free energy (2.3) can be rewritten in the form

$$2\lambda \tilde{f}_\infty (t, \lambda|d) = \lambda a(\phi, d) - (\phi + 2d) - \lambda s(\phi, b, d), \tag{3.3}$$

where

$$a(\phi, d) = \frac{1}{2\sqrt{\pi}} \int_0^\infty \frac{dx}{x^{3/2}} \exp (-x\phi) \left[1 - (\exp (-2x) I_0(2x))^d \right] + \sqrt{\phi}, \tag{3.4}$$

$$s(\phi, b, d) = 2 \int_0^\infty \frac{dx}{x} (4\pi x)^{-(d+1)/2} \exp\left(-x\phi\right) R\left(\frac{\pi^2}{xb^2}\right), \tag{3.5}$$

$$R(x) = \sum_{m=1}^\infty \exp\left(-xm^2\right), \tag{3.6}$$

$I_0(x)$ is a modified Bessel function, and ϕ in (3.3) is the solution of the corresponding spherical field equation that follows from (3.3) by requiring that the first partial derivative of the right hand side of (3.3) with respect to ϕ is zero. The above expressions are valid for *any* d.

In the remainder we will consider the dimensions $d = 1, d = 2$ and $d = 4$. Then it can be shown that:

a) for $d = 1$

$$a(\phi, 1) \quad = \frac{1}{2} \; {}_2F_1\left(-\frac{1}{4}, \frac{1}{4}, 1, \frac{4}{(2+\phi)^2}\right) \sqrt{2 + \phi}, \tag{3.7}$$

$$s(\phi, b, 1) = -\frac{b}{2\pi^2} \sqrt{\phi} \sum_{m=1}^\infty m^{-1} K_1\left(\frac{2\pi m \sqrt{\phi}}{b}\right) \tag{3.8}$$

where ${}_2F_1$ is the hypergeometric function and $K_1(x)$ is the MacDonald function (second modified Bessel function).

b) for $d = 2$, and $\phi << 1$

$$a(\phi, 2) \quad \simeq a(0, 2) + \mathcal{W}_2(0)\phi - \frac{1}{6\pi}\phi^{3/2} \tag{3.9}$$

$$s(\phi, b, 2) = -\left(\frac{b}{2\pi}\right)^3 \left[\frac{\sqrt{\phi}}{b} \mathrm{Li}_2\left(\exp\left(-\frac{2\pi\sqrt{\phi}}{b}\right)\right)\right.$$

$$\left. + \frac{1}{2\pi}\mathrm{Li}_3\left(\exp\left(-\frac{2\pi\sqrt{\phi}}{b}\right)\right)\right] \tag{3.10}$$

where

$$\mathcal{W}_d(\phi) = \frac{1}{2(2\pi)^d} \int_{-\pi}^\pi dq_1 \ldots \int_{-\pi}^\pi dq_d \left(\phi + 2\sum_{i=1}^d (1 - \cos q_i)\right)^{-1/2}. \tag{3.11}$$

is a Watson type integral, $\mathcal{W}_2(0) \approx 0.3214$, and $\mathrm{Li}_n(x)$ is the polylogarithmic function.

c) for $d = 4$, and $\phi << 1$

$$a(\phi, 4) \simeq a(0, 4) + \mathcal{W}_4(0)\phi - \frac{1}{2}r\phi^2 + \frac{1}{30\pi^{3/2}}\phi^{5/2}, \tag{3.12}$$

$$s(\phi, b, 4) = -\left(\frac{b}{2\pi}\right)^5 \left[\frac{\phi}{b^2}\text{Li}_3\left(\exp\left(-\frac{2\pi\sqrt{\phi}}{b}\right)\right)\right.$$
$$\left. +\frac{3\sqrt{\phi}}{2\pi b}\text{Li}_4\left(\exp\left(-\frac{2\pi\sqrt{\phi}}{b}\right)\right) + \frac{3}{4\pi^2}\text{Li}_5\left(\exp\left(-\frac{2\pi\sqrt{\phi}}{b}\right)\right)\right] \tag{3.13}$$

where $\mathcal{W}_4(0) \approx 0.1891$ and

$$r = \int_0^\infty \sqrt{x}\left[\exp\left(-2x\right)I_0\left(2x\right)\right]^4 dx \approx 0.0677. \tag{3.14}$$

Now we have the basic expressions needed to analyze the behavior of the finite-temperature C-function as defined by Eq. (1.1).

4. The behavior of the C-function

4.1. The case $d = 1$

From Eqs. (3.3), (3.7) and (3.8) it is easy to see that the only nonanalyticity in the behavior of the free energy exists at $\phi = 0$. Then, for $0 < \phi << 1$ one obtains, after some algebra, that the C-function [15] can be written in the following scaling form

$$C(t, \lambda) = \frac{\sqrt{\pi/2}}{6}y_0^{1/4}\exp\left(-\sqrt{y_0}\right), \tag{4.1}$$

where the scaling variable is $y_0 = \lambda^2\phi_0/t^2$, and we have identified $v = \sqrt{gJ}$. Here

$$\phi_0 = 64\exp\left(-4\pi/\lambda\right) \tag{4.2}$$

is the Haldane gap type behavior of the solution of the corresponding spherical field equation for the zero temperature system. Such type of solution is very well known from different problems, e. g., one dimensional anharmonic crystal [12] and the quantum nonlinear $O(N)$ sigma model in the large N limit (see, e.g., [13], [14]). In deriving (4.1) we have been interested in such a behavior of the nonzero temperature system which approaches the corresponding zero-temperature behavior when $T \to 0$.

As it is clear from the above expressions, C is a positive and a monotonically increasing function of the temperature.

The behavior of the C-function for $d = 1$ is illustrated in Fig. 1.

4.2. The case $d = 2$

We are interested in the behavior of the C-function around and below the critical point only, i.e. $0 < \phi << 1$ [10], [11]. As it is well known the critical point is at $\lambda = \lambda_c = 1/\mathcal{W}_2(0) \approx 3.1114$ and $T = 0$. Then, taking into account in Eq. (1.1) that

$n(2) = \zeta(3)/(2\pi)$ and $v = \sqrt{gJ}$, for the C-function we obtain from Eqs. (3.9) and (3.10) that $C(t, \lambda) = X(x)$, where

$$X(x) = \frac{1}{\zeta(3)} \left[x(y - y_0) + \frac{1}{6} \left(y^{3/2} - y_0^{3/2} \right) \right.$$
$$+ \left. \sqrt{y}\mathrm{Li}_2 \left(\exp\left(-\sqrt{y}\right) \right) + \mathrm{Li}_3 \left(\exp\left(-\sqrt{y}\right) \right) \right], \tag{4.3}$$

with $x = \pi (1/\lambda - 1/\lambda_c) \lambda/t$. Here $y = y(x)$, and $y_0 = y_0(x)$ are solutions of the corresponding equations that follow from (4.3) by requiring that the first partial derivative of the r.h.s. of (4.3) with respect to y, and y_0, respectively, to be zero. These solutions are

$$\sqrt{y} = 2\mathrm{arcsh}\left(\frac{1}{2}\exp\left(-2x\right) \right), \tag{4.4}$$

and

$$\sqrt{y_0} = \begin{cases} -4x & , \quad \lambda > \lambda_c \\ 0 & , \quad \lambda \le \lambda_c \end{cases}. \tag{4.5}$$

The equation (4.3) determines the *exact scaling function of C* for the case $d = 2$. From the above equations one can see the different behavior of y in the three regions: i) *renormalized classical*, where y tends to zero exponentially fast as a function of x ($x \gg 1$) ii) *quantum critical*, where $y = O(1)$ (for $x = O(1)$) and iii) *quantum disordered*, where y diverges as $(4x)^2$ for $x \ll -1$; ($y \sim (\chi t^2)^{-1}$, where χ is the susceptibility of the system, see [10]). The location of these regions is depicted in Fig. 2.

The behavior of the C-function reflects the existence of these three regions.

When $\lambda < \lambda_c$ and $t \to 0$, from Eqs. (4.3) -(4.5) it follows that

$$C(t, \lambda) \simeq 1 - \frac{1}{4\zeta(3)} \exp\left[-4\pi \left(1 - \lambda/\lambda_c \right) t^{-1} \right]. \tag{4.6}$$

One explicitly observes the exponentially small corrections to the limit value of $C = 1$ (at $t = 0$) that corresponds to massless bosons in d dimensions [3], [4].

At $\lambda = \lambda_c$, $C(t, \lambda)$ simplifies, by using the Sachdev's identity [4], and becomes

$$C(t, \lambda_c) = \frac{4}{5}. \tag{4.7}$$

This universal *rational* number [16] has been derived for the first time for the quantum nonlinear $O(N)$ sigma model in the limit $N \to \infty$ [4]. It demonstrates that at the quantum critical point $\lambda = \lambda_c$ the C-function does not depend on the temperature. The difference from the corresponding results in Ref. [3] (compare Fig. 3 in [3] with Fig. 3 in this article) is due to the fact that terms proportional to the difference between y and y_0 in (4.3) have been neglected there. The above is justified when $y \gg 1$ (then y and y_0 are exponentially close to each other). The analysis of the corresponding equation

shows that the last happens when $x << -1$ (i.e. $\lambda > \lambda_c$) where $y \sim y_0 \sim (4x)^2$, which is the case of the quantum disordered region. In this case it is easy to see that

$$C(t, \lambda) \simeq \frac{4\pi}{\zeta(3)} \frac{|1 - \lambda/\lambda_c|}{t} \exp\left[4\pi \left(1 - \lambda/\lambda_c\right) t^{-1}\right],\tag{4.8}$$

i.e. C approaches zero exponentially fast in terms of the scaling parameter x. The behavior of the C-function in this case is that one of massive free bosons [3].

Let us consider now the monotonicity of the C-function. From (4.3) it follows that

$$\frac{\partial C(t, \lambda)}{\partial t} = -\frac{\pi}{\zeta(3)} \left(y - y_0\right) \left(1 - \lambda/\lambda_c\right) t^{-2}.\tag{4.9}$$

Since $y > y_0$, we conclude that C is a monotonically increasing function of the temperature for $\lambda > \lambda_c$, and monotonically decreasing function for $\lambda < \lambda_c$. Within exponentially small in temperature corrections this result coincides, in fact, with the corresponding one for the QNLσM in the limit $N \to \infty$ [3].

The above results for the behavior of the C-function are illustrated in Fig. 3.

Finally, it seems worthwhile to mention that, as follows from Eq. (4.3), *the C-function is monotonically increasing function of the scaling variable x* (see Fig. 4) *for any value of t ($\lambda/t \gg 1$).*

4.3. The case $d = 4$

For this case, taking into account that in Eq. (1.1) $n(4) = 3\zeta(5)/(2\pi)^2$ and $v = \sqrt{gJ}$, for the C-function we obtain from Eqs. (3.12) and (3.13) that $C(t, \lambda) = X(x, \lambda/t)$, where

$$X(x, \lambda/t) = \frac{(2\pi)^2}{3\zeta(5)} \left[x(y - y_0) + \frac{1}{4}r\left(y^2 - y_0^2\right)\frac{\lambda}{t}\right. \tag{4.10}$$
$$\left. + \frac{1}{(2\pi)^2}\left[y\mathrm{Li}_3\left(\exp\left(-\sqrt{y}\right)\right) + 3\sqrt{y}\mathrm{Li}_4\left(\exp\left(-\sqrt{y}\right)\right) + 3\mathrm{Li}_5\left(\exp\left(-\sqrt{y}\right)\right)\right]\right]$$

with

$$x = \frac{1}{2}\left(\frac{\lambda}{t}\right)^3\left(\frac{1}{\lambda} - \frac{1}{\lambda_c}\right),\tag{4.11}$$

and $\lambda_c = 1/\mathcal{W}_4(0) \approx 5.2882$. Here $y \geq 0$, and $y_0 \geq 0$ are solutions of the corresponding equations that follow from (4.10) by requiring that the first partial derivative of the r.h.s. of (4.10) with respect to y, and y_0, respectively, to be zero. This leads to the following equation for y

$$x = -\frac{1}{2}r\frac{\lambda}{t}y + \frac{1}{2(2\pi)^2}\left[\sqrt{y}\mathrm{Li}_2\left(\exp\left(-\sqrt{y}\right)\right) + \mathrm{Li}_3\left(\exp\left(-\sqrt{y}\right)\right)\right].\tag{4.12}$$

It is easy to see that, for a given t and λ, the solution of the above equation, if it exists, is unique. For y_0 we get

$$y_0 = \begin{cases} -\tilde{x} = -\left(2t\right)/\left(r\lambda\right)x, & \lambda > \lambda_c \\ 0, & \lambda \le \lambda_c \end{cases}. \qquad (4.13)$$

One observes that in the most general case the function X, given by Eq. (4.10), could not be recast in a scaling form. However, as we will see below, the last is possible in some subregions of the $\lambda - t$ plane. We recall that the susceptibility χ of the system is proportional to y^{-1} (if $y \ne 0$ [18], [19]), which leads to the conclusion that a nonzero-temperature phase transition exists at a given $t_c = t_c(\lambda)$, where $t_c(\lambda)$ is given by the equation

$$t_c(\lambda) = \lambda \left[\frac{(2\pi)^2}{\zeta\left(3\right)} \left(\frac{1}{\lambda} - \frac{1}{\lambda_c} \right) \right]^{1/3}, \qquad (4.14)$$

(at $t = t_c(\lambda)$ one has $y = 0$, and $y = 0$ also for $t < t_c(\lambda)$). As for the $d = 2$ case three principal different regimes exist: i) *renormalized classical (*where y tends to zero exponentially fast as a function of λ/t), ii) *quantum critical* (where y tends to zero algebraically as a function of λ/t or $y = O(1)$), and iii) *quantum disordered (*where $y >> 1$). In order to describe the behavior of the C-function below we analyze these three regimes.

A) Let us first suppose that $y << 1$. Then Eq. (4.12) becomes

$$\left(\frac{\lambda}{t}\right)^3 \left[\frac{1}{\lambda} - \frac{1}{\lambda_c} - \frac{\zeta(3)}{(2\pi)^2} \left(\frac{t}{\lambda}\right)^3 \right] = \frac{1}{(4\pi)^2} y \ln\left(y/e\right) - r\frac{\lambda}{t}y. \qquad (4.15)$$

Obviously, there are two subregimes *a)* when the first term in the right hand side dominates and *b)* when the second one dominates. The borderline between them is given by

$$\frac{1}{2r}\left(\frac{\lambda}{t}\right)^2 \left[\frac{1}{\lambda_c} - \frac{1}{\lambda} + \frac{\zeta(3)}{(2\pi)^2} \left(\frac{t}{\lambda}\right)^3 \right] = \exp\left[-(4\pi)^2 r\frac{\lambda}{t} + 1 \right]. \qquad (4.16)$$

In the $\lambda - t$ plane Eq. (4.16) determines a line $t^*(\lambda)$ that is exponentially close to (as a function of λ/t) the line $t_c(\lambda)$. At $t^*(\lambda)$ the solution of Eq. (4.15) is

$$y \sim \exp\left[-(4\pi)^2 r\frac{\lambda}{t} \right], \qquad (4.17)$$

whereas $y = 0$ at $t_c(\lambda)$. We conclude that the *renormalized classical regime* is observed for parameters of the system laying in the $\lambda - t$ plane between $t_c(\lambda)$ and $t^*(\lambda)$. In this regime one could neglect the second term in the r.h.s. of (4.10) which leads to a scaling form of the C-function with a scaling variable x, defined in (4.11). At $t^*(\lambda)$ the C-function could not be rewritten in a scaling form. In the remainder we will see that

another scaling variable $\tilde{x}$ can be defined for the region to the right of $t^*(\lambda)$. This is due to the fact that in this region the first term in the r.h.s. of Eq. (4.12) is of the leading order. Indeed, this is true not only for the case *b)* but also for the cases *B)*, when $y = O(1)$, and *C)*, when $y \gg 1$, which cases are to be considered below.

Before passing to the consideration of case *B)* let us note that the further inspection of Eq. (4.15) for the case *b)* leads to the conclusion that there exists a crossover line

$$t_s(\lambda) = \lambda \left[\frac{(2\pi)^2}{\zeta(3)} \left(\frac{1}{\lambda_c} - \frac{1}{\lambda} \right) \right]^{1/3}, \tag{4.18}$$

between two regimes where $\chi(t,\lambda) \sim t^{-3}$ and $\chi(t,\lambda) \sim t^{-2}$, respectively. This curve is symmetric to the curve $t_c(\lambda)$ with respect to the line $\lambda = \lambda_c$. Let us turn now to the case

B) $y = O(1)$. Then, since $t << 1$, Eq. (4.12) becomes extremely simple and, up to the leading order coincides with the corresponding equation for the zero-temperature system (see Eq. (4.13)). Its solution is

$$y = \frac{1}{r} \left(\frac{\lambda}{t} \right)^2 \left(\frac{1}{\lambda_c} - \frac{1}{\lambda} \right) \equiv -\tilde{x}. \tag{4.19}$$

Since $\chi(t,\lambda) \sim (yt^2)^{-1}$ and $y = O(1)$, one concludes that in this regime $\chi(t,\lambda) \sim t^{-2}$. In a given sense a formal curve $t_1(\lambda)$ in the $\lambda - t$ plane which borders the region in which $y = O(1)$ can be obtained by simply setting $y = 1$ in Eq. (4.19). Summarizing the results from *A)* and *B)* we are lead to the conclusion that the *quantum critical regime* is observed for values of the parameters t and λ laying in the $\lambda - t$ plane between the curves $t^*(\lambda)$ and $t_1(\lambda)$. We see also that in this region $X(t,\lambda) = X(\tilde{x})$, i.e. the scaling is restored with the new scaling variable $\tilde{x}$. We pass now to the case

C) $y >> 1$. Then one formally receives the same solution as given by Eq. (4.19) but now $\chi(t,\lambda) \sim (1/\lambda_c - 1/\lambda)^{-1}$, i.e. it does not depend on t up to exponentially small in λ/t corrections. We conclude that the region of parameters in the $\lambda - t$ plane below $t_1(\lambda)$ determines the *quantum disordered* region. Again, as in B), the scaling variable of C is $\tilde{x}$.

The above results are summarized in Fig. 5.

The existence of the regions of thermodynamic parameters determined above is reflected by the corresponding behavior of the C-function given by Eq. (4.10).

First, for $t \leq t_c(\lambda)$ (i.e. under the existence of a long-range order in the system), since $y = y_0 = 0$, one immediately obtains from (4.10) that $C = 1$ [20].

Further, for $t > t_c(\lambda)$ taking into account Eqs. (4.13) and (4.12), it is easy to see that

$$\frac{\partial C(t,\lambda)}{\partial t} = -\frac{2\pi^2 r}{\zeta(5)} \frac{\lambda}{t^2} (y - y_0) \left[\tilde{x} + \frac{1}{6} r (y + y_0) \right]. \tag{4.20}$$

Since $y > y_0 > 0$ for the considered region of parameters λ and t, the above equation leads us to the conclusion that for any $\lambda < \lambda_c$ the C-function is a *monotonically decreasing function of the temperature*. The same is true also for $\lambda = \lambda_c$ and $t \neq 0$. From the analysis of the solutions of the equations for y and y_0 it becomes clear that for a fixed λ and $t > t_s(\lambda)$ the *leading-order* form for both y and y_0 is $y = y_0 = -\tilde{x}$ (if one takes into account next-to-the-leading order terms then, of course, $y > y_0$). Setting the above expressions for y and y_0 in the rectangular brackets in Eq. (4.20), we conclude that C is a *monotonically increasing function of t* for any $t > t_s(\lambda)$. It is clear that somewhere between $\lambda = \lambda_c$ and $t_s(\lambda)$ the derivative of the C-function changes its sign, i.e. there is a line $t_{st}(\lambda)$ of stationary points $\partial C(t,\lambda)/\partial t = 0$. One can see that

$$t_{st}(\lambda) = \lambda \left[\frac{(4\pi)^2}{\zeta(3)} \left(\frac{1}{\lambda_c} - \frac{1}{\lambda} \right) \right]^{1/3} . \tag{4.21}$$

Since the point $\lambda = \lambda_c, t = 0$ lies on $t_{st}(\lambda)$ and at it $C = 1$, we conclude that $C = 1$ at the whole line $t_{st}(\lambda)$. It is clear now that C is a *monotonically increasing* function of t for $t < t_{st}(\lambda)$. It is a *monotonically decreasing* function of t for $t > t_{st}(\lambda)$ as well as for any (small) t if $\lambda < \lambda_c$. These results are summarized in Fig. 6.

5. Finite-size scaling interpretation

It is interesting to interpret the bulk critical behavior of the C-function in the context of the finite-size scaling (FSS) theory by introducing a finite "temporal" dimension $L_\tau = \lambda/t$. Then, taking into account: *i)* the dimensional crossover rule that connects the properties of a given d-dimensional quantum system and those ones of the corresponding $(d+1)$-dimensional classical one with the mapping $d \to \mathrm{d}+1, L \to L_\tau, t \to \lambda$, *ii)* the Privman and Fisher [24] hypothesis for the free energy of a finite classical system when the hyperscaling holds (i.e. between the lower d_l and the upper d_u critical dimensions of the system), one could make the statement that the free energy f_∞ of a quantum system with dimensionality $d_l < d < d_u$ should have the form

$$\left[f_\infty\left(t, \lambda | d\right) - f_\infty\left(0, \lambda | d\right) \right] / (T L_\tau)^{-1} = L_\tau^{-(d+1)} Y\left(L_\tau / \xi\left(0, \lambda\right)\right), \tag{5.1}$$

where $\xi(0,\lambda)$ is the correlation length of the zero-temperature system, and Y is a *universal function*. We remind that in order to have no nonuniversal prefactor in front of Y for classical systems one considers $\tilde{f}_\infty = \beta f_\infty$, instead of f_∞ itself. The normalization of the free energy in (5.1) simply follows by our choice of L_τ. For the model considered here the inspection of Eq. (4.3) shows that the hypothesis (5.1) is indeed valid with the standard scaling variable $L_\tau / \xi(0,\lambda) \equiv x = \pi(1/\lambda - 1/\lambda_c)\lambda/t$ and $C(t,\lambda) = X(x)$ $= -Y(x)/n(d)$. It is interesting that despite the lack of hyperscaling at $d_l = 1$ the C-function again can be written as a function of $L_\tau / \xi(0,\lambda)$ (see Eq. (4.1)), if one

identifies $\xi(0,\lambda) = \phi_0^{-1/2}$, where ϕ_0 is given by Eq. (4.2). The case $d = 4 > d_u$ is much more interesting due to the lack of hyperscaling. In the most general case the C-function could not be even recast in a FSS form (see Eq.(4.10)). The last is possible exponentially close (in L_τ) to the line of finite-temperature phase transitions $t_c(\lambda)$, where the modified scaling variable is $2x = L_\tau [L_\tau/\xi(0,\lambda)]^2$ (see Eq. (4.11)). The standard scaling variable $\tilde{x} = L_\tau/\xi(0,\lambda)$ is restored only for parameters to the right of the curve $t^*(\lambda)$ in the $\lambda - t$ plane (see Eq. (4.19) and the comments connected with it). This change of scaling variables from x to $\tilde{x}$ is a new point within FSS theory. Normally one observes modified FSS (see [25], [26]) above d_u due to the existence of a dangerous irrelevant variables in the system. On the other hand, considering the $d = 5$ dimensional spherical model film Barber and Fisher, as early as in 1973 [27], stated that the scaling variable should be the standard one, i.e. $L_\tau/\xi(0,\lambda)$ in our notations. The above results resolve this seeming contradiction: the scaling variable has to be modified very close to the phase boundary, but is the standard one a bit away of it. The physical reasoning for that difference is the existence in the system of a temperature driven phase transition in addition to the quantum one with respect to λ at $t = 0$. To our knowledge all other examples considered previously in the literature of modified FSS concerns finite systems with no (sharp) phase transition in it.

6.   Concluding remarks

One generally expects that the C-function increases monotonically when the quantum fluctuations "dominate" [3]. The real meaning of the term "dominate" turns out to be quite subtle, as we have demonstrated in the current article. In fact, we have shown that the region where the C-function remains monotonically increasing (as a function of temperature) and the quantum critical region do essentially intersect but do *not* coincide (see Fig. 2). This is one of the results of the present work. The question of where one should look for and what should be understood as domination of quantum fluctuations is, indeed, very intriguing. It is a part of the more general problem of a *quantitative* description of the interplay of the quantum and critical fluctuations. There exist different views on that issue. The standard one [21], [22] is based on the "ratio" between the correlation length and the length of De Brougle. Another possible approach can be based on the behavior of the C-function [3], [5]. Furthermore, there is an approach based on the algebra of critical fluctuation operators, due to Verbeure and Zagrebnov [23], where a measure of the "degree of criticality" is introduced in a mathematically rigorous way.

In the present work we investigated the behavior of the C-function for $d = 1, 2, 4$. The case $d = 1$ represents the situation with no phase transition and strong quantum fluctuations, $d = 2$ – the one when a quantum critical point appears at $T = 0$, and

$d = 4$ – when there is a line of classical critical points ending up with a zero temperature (quantum) critical point. In fact, these are the most typical cases on which the attention in the literature is focused.

Case $d = 1$. As it is to be expected on general grounds, the C-function increases monotonically as a function of temperature (see Fig. 1). This reflects the fact that the quantum fluctuations are strong enough (as it is clear from Eq. (4.2) one cannot consider λ as a small parameter) and the lack of a critical point. The C-function obtained here coincides with the C-function of the massive free bosons (for $d = 1$) with mass $\sqrt{\phi_0}$, because of the exponentially small difference then between ϕ and ϕ_0, i.e. one can consider ϕ as a fixed parameter in (3.7) and in (3.8). The general case (for any d) of free massive bosons actually follows from (3.3) - (3.6) by considering ϕ there as a fixed parameter connected to the mass m of bosons ($\phi \sim m^2$). For the last case it is trivial to check that the corresponding C-function is that one obtained in [3] (see Eq. (3.2) there).

Case $d = 2$. Fig. 2 shows the phase diagram for our model which coincides with the phase diagram of the $d = 1$ quantum Ising model, as well as with the nonlinear $O(n)$ sigma model in the limit $n \to \infty$, see, e.g., Sachdev [21]. As a function of the temperature C is monotonically increasing for λ above λ_c, equals 4/5 at $\lambda = \lambda_c$ (and then C does not depend on t) and is monotonically decreasing for λ below λ_c (see Fig. 3). The lack of overall monotonicity with respect to the temperature is due to the crossover from classical to quantum behavior. It is clear, that one indeed can consider *monotonicity of C as a measure* of the role the corresponding fluctuations are playing in a given region of parameters. It is interesting that C changes its monotonicity, in fact, in the *middle* of the quantum critical region. Finally, we note that it is nevertheless possible to find a (nontrivial) variable, with respect to which the C-function is monotonic in the whole $t-\lambda$ plane (see Fig. 4). This variable is the scaling variable $x = \pi \left(1/\lambda - 1/\lambda_c\right) \lambda/t$.

Case $d = 4$. The existence of a line of non-zero temperature critical points modifies drastically the corresponding picture in comparison with the $d = 2$ case. Now a line of stationary points $t_{st}(\lambda)$ appears (see Fig. 5) which "starts" from $(\lambda = \lambda_c, t = 0)$ and lies to the left of $t_s(\lambda)$. To the left of $t_{st}(\lambda)$, C is a non-increasing function of the temperature (see Fig. 6). For $\lambda < \lambda_c$ and $t < t_c(\lambda)$ one has $C = 1$, whereas within the region between $t_c(\lambda)$ and $t_{st}(\lambda)$ the C-function is monotonically decreasing as a function of the temperature. To the right of $t_{st}(\lambda)$ the C-function becomes monotonically increasing function of the temperature, being zero at the $t = 0$, $\lambda > \lambda_c$ line. At the lines $t_c(\lambda)$ and $t_{st}(\lambda)$ the C-function reaches its maximum value, i.e. it becomes $C = 1$. Finally, we would like to mention that, similar to the case $d = 2$, it is possible to find two nontrivial parameters such that with respect to both of them the C-function is monotonically increasing. Such are, e.g., the parameters x and λ/t (see Eq. (4.10) and take into account that $y > y_0$).

Comparing the behavior of the C-function for $d = 1$, $d = 2$ and $d = 4$ we conclude that

a) For $d = 1$ for any fixed λ we have a monotonically increasing with temperature C-function.

b) For $d = 2$ the above is true only for $\lambda > \lambda_c$.

c) For $d = 4$ the C-function is a monotonically increasing function of t for $\lambda > \lambda_c$ and t *small enough*. The monotonicity of the C-function does not change by increasing t *only* for $\lambda = \lambda_c$.

So, the region in the parametric space where C remains monotonically increasing with t becomes smaller when d increases. We explicitly see the crucial role the dimensionality d and the existence of phase transition, which appears upon increasing d, play in the behavior of the C-function as a function of t. Nevertheless, for any d one can find nontrivial variable(s), function(s) of the temperature and the parameter controlling the quantum fluctuations, in terms of which C is a monotonically increasing function of its variable(s). In close vicinity of the quantum critical point the C-function is given by a universal scaling function which properties can be interpreted in terms of FSS that has to be modified for $d = 4$.

Acknowledgments

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[20] Note that this result does depend only on the existence of long-range order in the system (then $y = y_0 = 0$) and not on the dimensionality d. From Eqs. (1.1) and (3.3) - (3.5) and the identity

$$\Gamma\left(\frac{s}{2}\right)\pi^{-s/2}\zeta\left(s\right) = \int_0^\infty R\left(\pi s\right)x^{s/2}\frac{dx}{x}, \qquad \mathrm{Re}\,s > 1,$$

we get $C = 1$.

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Figure Captions

Fig. 1. The bold line $t_{LT} = 8\lambda \exp\left(-2\pi/\lambda\right)$ borders from above the region in the $t-\lambda$ plane where the expression for the 1-d C-function, given by Eq. (4.1), is valid. The symbol $C \uparrow$ means that C increases in whole that region starting form $C = 0$ at $t = 0$.

Fig. 2. The phase diagram of the model and crossovers for the case $d = 2$ as a function of t and the normalized quantum parameter λ. One distinguishes renormalized classical, quantum critical and quantum disordered regions. Long-range order is present only at $t = 0$ for $\lambda < \lambda_c$.

Fig. 3. The behavior of the 2-d C-function is illustrated for $\lambda = 1.5\lambda_c$, $\lambda = \lambda_c$ and $\lambda = 0.5\lambda_c$.

Fig. 4. The behavior of 2-d C as a function of the scaling parameter $x = \pi\left(1/\lambda - 1/\lambda_c\right)\lambda/t$.

Fig. 5. The phase diagram of the model and crossovers for the case $d = 4$. Long-range order exists below the line t_c. The line t_{st} is the locus of points in the thermodynamic space where $\partial C/\partial t = 0$. The other lines denote crossovers between different regimes which are described in the text.

Fig. 6. The monotonic behavior of C as a function of temperature is shown. The symbols $C \uparrow$ ($C \downarrow$) mean that C is a monotonically increasing (decreasing) function of the temperature. The number in ellipses show the value of C at the corresponding line. In whole long-range order region, i.e. below the line t_c, $C = 1$.

t

t_{st}

t_c

$C=1$

$C\downarrow<1$

$C=1$

$C\uparrow<1$

t_s

$C=1$

$C\uparrow<1$

$C=1$

$C=0$

$\lambda-\lambda_c$